Morin's Classical Mechanics

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1.1
The relevant variables are G (gravitational constant), M (mass of the planet), m (mass of the object), and R (distance from the planet). The escape velocity cannot depend on mass of the object because Newton's law of gravity takes the form of $\frac{GMm}{r^2}$, and because of $F = ma$, acceleration due to gravity divides that by m, leaving $\frac{GM}{r^2}$, which is independent of m. Thus, the variables this can depend on are G, M, and R. Therefore,

$$(\frac{L^3}{MT^2})^a (M)^b (L)^c = (\frac{L}{T})$$

Focusing on T, we have

$$(\frac{1}{T^2})^a = (\frac{1}{T})$$

and thus $a = \frac{1}{2}$. Plugging this back in we get

$$(\frac{L^3}{MT^2})^{\frac{1}{2}} (M)^b (L)^c = (\frac{L}{T})$$

Now focusing on M we have

$$(\frac{1}{M})^{\frac{1}{2}} (M)^b = 1$$

so $b = \frac{1}{2}$. Finally focusing on L we have

$$(L^3)^{\frac{1}{2}} (L)^c = L$$

and thus $c = -\frac{1}{2}$. Thus escape velocity must be

$$\sqrt{\frac{GM}{R}}$$

up to numerical factors.
1.2
The relevant variables are M, m and ℓ. η is dimensionless, so we have

$$(M)^a (M)^b (L)^c = 1$$

. Therefore trivially $a = -b$ and $c = 0$, so it does not depend on ℓ.
1.3
The relevant variables are ρ and B, so we have

$$(\frac{M}{L^3})^a (\frac{M}{LT^2})^b = \frac{L}{T}$$

This gives $a = -\frac{1}{2}$ and $b = \frac{1}{2}$, so speed must be

$$\sqrt{\frac{B}{\rho}}$$

up to numerical factors.
1.4
The relevant variables are R, ρ and G, so we have

$$(L)^a(\frac{M}{L^3})^b(\frac{L^3}{MT^2})^c = \frac{1}{T}$$

Looking at T we have

$$(\frac{1}{T^2})^c = \frac{1}{T}$$

so $c = \frac{1}{2}$. Looking at L we have

$$\frac{L^{3c}}{L^{\frac{3}{2}}} = 1$$

so b must equal $\frac{1}{2}$ and a must equal 0. Thus frequency must be

$$\sqrt{\rho G}$$

up to numerical factors.
1.5
a) The relevant variables are m, V, and b, however the units of b depend on n. bv^n must have the units of force, so we have

$$(b)(\frac{L}{T})^n = \frac{ML}{T^2}$$

$$b = (\frac{ML}{T^2})(\frac{T}{L})^n$$

$$b = \frac{MT^{n-2}}{L^{n-1}}$$

Plugging this in we have

$$(M)^a(\frac{L}{T})^b(\frac{MT^{n-2}}{L^{n-1}})^c = T$$

Focusing on T this is

$$(\frac{1}{T})^b(T^{n-2})^c = T$$

thus

$$1 + b = c(n-2)$$

Focusing on M this is

$$(M)^a(M)^c = 1$$

thus

$$a = -c$$

Focusing on L this is

$$(L)^b(\frac{1}{L^{n-1}})^c = 1$$

thus

$$b = c(n-1)$$

We have

$$1 + b = cn - 2c$$

$$b = cn - c$$

Subtracting we get that $c = -1$, thus $a = 1$ and $b = 1 - n$. Thus the stopping time is

$$(\frac{m}{b})(V^{1-n})$$

up to numerical factors. b) Plugging everything in we have

$$(M)^a(\frac{L}{T})^b(\frac{MT^{n-2}}{L^{n-1}})^c = L$$

Focusing on T this is

$$(\frac{1}{T})^b(T^{n-2})^c = 1$$

thus

$$b = c(n-2)$$

Focusing on M this is

$$(M)^a(M)^c = 1$$

thus

$$a = -c$$

Focusing on L this is

$$(L)^b(\frac{1}{L^{n-1}})^c = 1$$

thus

$$b + 1 = c(n-1)$$

We have

$$b = cn - 2c$$

$$1 + b = cn - c$$

Subtracting we get that $c = 1$, thus $a = -1$ and $b = n - 2$. Thus the stopping distance is

$$(\frac{b}{m})(V^{n-2})$$

up to numerical factors.

1.6

$\frac{gh^2}{v^2}$ and $\sqrt{\frac{v^2h}{g}}$ go to zero for h=0, which is not right because the ball can still travel up to $\frac{v^2}{g}$ from $h = 0$. $\frac{v^2}{g}$ is incorrect because it has no h dependence, which the correct solution obviously would have. $\frac{\frac{v^2}{g}}{1-\frac{2gh}{v^2}}$ is wrong because if you increase h too much it becomes negative, which is nonsensical in this context. $\frac{v^2}{g}(1 + \frac{2gh}{v^2})$ is wrong because this simplifies to $\frac{v^2}{g} + 2h$, which is 2h at $v = 0$, not zero. Therefore, $\frac{v^2}{g}\sqrt{1 + \frac{2gh}{v^2}}$ must be correct by process of elimination.

1.7

https://azgeorgis.net/rmg/morin/1-7.html

2.1

Balancing forces on a piece of the rope gives

$$T(\ell + d\ell) = T(\ell) + \rho d\ell g$$

thus

$$dT = \rho g d\ell$$

thus

$$T(\ell) = \rho g \ell$$

2.2

The normal force on the block is $mg\cos(\theta)$ perpendicular to the plane, so the horizontal component of the normal force is $mg\sin(\theta)\cos(\theta)$. Likewise, the friction force is just large enough to balance the gravitational force, so it is $\mu mg\sin(\theta)$, thus its horizontal component is $\mu mg\cos(\theta)\sin(\theta)$. These are maximized when their derivatives are zero, so we have

$$\frac{d}{d\theta}\cos(\theta)\sin(\theta) = 0$$

$$\cos^2(\theta) - \sin^2(\theta) = 0$$

$$\cos(\theta) = \sin(\theta)$$

$$\theta = \frac{\pi}{4}$$

2.3

The chain is in the shape of a function $y(x)$ with both endpoints equal. By trigonometry, he angle between the chain and the horizontal is

$$\tan^{-1}(\frac{dy}{dx})$$

The length of a chunk of the chain is

$$\sqrt{\frac{dy}{dx}^2 + 1}$$

The force of gravity on a chunk of the chain is $g\lambda\sqrt{\frac{dy}{dx}^2 + 1}dx$, therefore the component of this parallel to the tube is

$$g\sin\left(\arctan\left(\frac{dy}{dx}\right)\right)\lambda\sqrt{\frac{dy}{dx}^2 + 1}dx$$

which is equivalent to

$$\frac{g\lambda\frac{dy}{dx}\sqrt{\frac{dy}{dx}^2 + 1}}{\sqrt{\frac{dy}{dx}^2 + 1}}dx$$

$$g\lambda\frac{dy}{dx}dx$$

Integrating to obtain the force over the whole chain we get

$$g\lambda \int_0^L \frac{dy}{dx}\, dx$$

which is by the fundamental theorem of calculus just the change in the function over that interval, but the function has both endpoints at the same height, so it is just 0.

2.4

a) Horizontal $F = ma$ equation:

$$F\cos(\theta) = F_{norm}$$

Vertical $F = ma$ equation:

$$Mg = F\sin(\theta) + \mu F_{norm}$$

Substituting and solving:

$$Mg = F\sin(\theta) + F\mu\cos(\theta)$$

$$F = \frac{Mg}{\sin(\theta) + \mu\cos(\theta)}$$

b) This is maximized when $\frac{dF}{d\theta} = 0$, so we have

$$\frac{Mg(\mu\sin(\theta) - \cos(\theta)}{(\mu\cos(\theta) + \sin(\theta))^2} = 0$$

Because the denominator cannot be infinity, the only way this can equal zero is if the numerator is zero, so

$$Mg(\mu\sin(\theta) - \cos(\theta) = 0$$

$$\mu\sin(\theta) - \cos(\theta) = 0$$

$$\cot(\theta) = \mu$$

$$\theta = \arctan(\frac{1}{\mu})$$

The corresponding minimum F is

$$F = \frac{Mg}{\sin(\arctan(\frac{1}{\mu})) + \mu\cos(\arctan(\frac{1}{\mu}))}$$

$$F = \frac{Mg}{\cos(\arctan(\mu)) + \mu\sin(\arctan(\mu))}$$

$$F = \frac{Mg}{\frac{1}{\sqrt{\mu^2+1}} + \frac{\mu^2}{\sqrt{\mu^2+1}}}$$

$$F = \frac{Mg}{\sqrt{\mu^2+1}}$$

c) The minimum θ would be when F goes to infinity, so

$$\sin(\theta) + \mu\cos(\theta) = 0$$

$$\tan(\theta) = -\mu$$

$$\theta = -\arctan(\mu)$$

2.5
The force of gravity on an infinitesimal chunk of the rope is

$$\rho g \sin(\theta)\, dx$$

The friction force satisfies

$$F_{frict} \leq \mu \rho g \cos(\theta)\, dx$$

For the minimum tension, the friction force counteracts gravity,so f the friction force is greater than the gravitational force, there will be zero tension at the top of the rope. This implies

$$\mu \rho g \cos(\theta)\, dx \geq \rho g \sin(\theta)\, dx$$

$$\mu \cos(\theta) \geq \sin(\theta)$$

$$\mu \geq \tan(\theta)$$

If $\mu < \tan(\theta)$, then there will be a minimum tension force of

$$\rho g \sin(\theta) L - \mu \rho g \cos(\theta) L$$

On the other hand for the maximum tension, the friction acts with gravity, so we have a maximum tension force of

$$\rho g \sin(\theta) L + \mu \rho g \cos(\theta) L$$

2.6
a) A small chunk of the rope applies a force to the disk of

$$2T sin(d\theta/2)$$

which at small angles is approximately equal to

$$T\, d\theta$$

The vertical component of this is

$$T \sin(\theta)\, d\theta$$

Integrating this from 0 to π we get

$$\int_0^\pi T \sin(\theta)\, d\theta = 2T$$

so

$$2T = Mg$$

$$T = \frac{Mg}{2}$$

The normal force per unit length is

$$\frac{T\pi}{\pi R} = \frac{T}{R}$$

b) For the minimum tension force the friction force should always counteract the tension. We have for a chunk of rope

$$dT \leq \mu T d\theta$$

so at the bottom

$$T \geq T_0 e^{\frac{\mu \pi R}{2}}$$

Since T_0 is $\frac{Mg}{2}$, the minimum tension at the bottom is

$$\frac{Mg}{2} e^{\frac{\mu \pi R}{2}}$$

2.7

For each shape, balancing forces on the object gives

$$2F_{norm} \sin(\theta) = mg$$

and balancing forces on the circles gives

$$F_{norm} \cos(\theta) = F$$

Solving we obtain

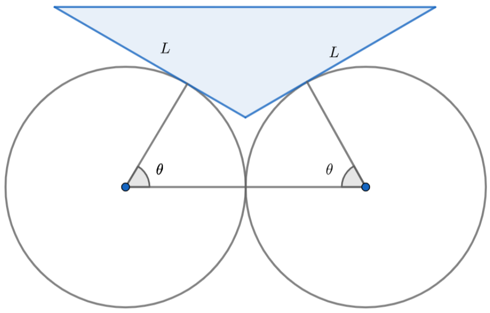
$$F = \frac{mg \cos(\theta)}{2 \sin(\theta)}$$

Since $m = \sigma A$ this is

$$F = \frac{\sigma A g \cos(\theta)}{2 \sin(\theta)}$$

The only thing remaining is to find A in terms of L and θ for each shape.

a)



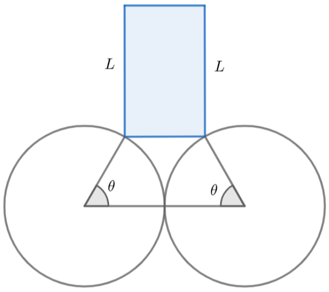
If you continue the radii of the circle until they touch the top side of the triangle, an isosceles trapezoid is formed with the bottom two angles as θ, therefore, the top two angles must be $\frac{\pi}{2} - \theta$. Thus, we can obtain the angles of the isosceles triangle to be $\frac{\pi}{2} - \theta$, $\frac{\pi}{2} - \theta$, and 2θ. The base is therefore $2L\sin(\theta)$, and the height is $L\cos(\theta)$, so the area is $L^2 \cos(\theta) \sin(\theta)$. Thus the F is

$$\frac{L^2 \cos(\theta) \sin(\theta) \sigma g \cos(\theta)}{2 \sin(\theta)}$$

$$\frac{L^2 \cos(\theta)^2 \sigma g}{2}$$

This value has a minimum of 0 at $\theta = \frac{\pi}{2}$.

b)



The width of the rectangle is

$$2R - 2R\cos(\theta)$$

$$2R(1 - \cos(\theta)$$

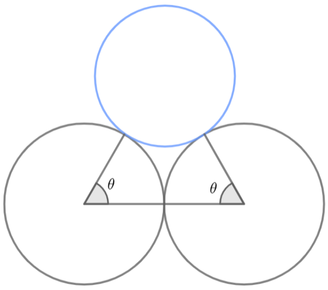
Thus the area is

$$2RL(1 - \cos(\theta))$$

and the force is

$$F = \frac{mgRL(1 - \cos(\theta))\cos(\theta)}{\sin(\theta)}$$

c)



The radius of the small circle is set by the equation

$$2R = 2(R + r)\cos(\theta)$$

$$R(1 - \cos(\theta)) = r\cos(\theta)$$

$$r = \frac{R(1 - \cos(\theta))}{\cos(\theta)}$$

Thus the area is

$$\pi(\frac{R(1 - \cos(\theta))}{\cos(\theta)})^2$$

Plugging that back in we get

$$F = \pi(\frac{R(1 - \cos(\theta))}{\cos(\theta)})^2 \frac{\sigma g\cos(\theta)}{2\sin(\theta)}$$

$$F = \frac{\pi\sigma g R^2(1 - \cos(\theta))^2}{2\sin(\theta)\cos(\theta)}$$

2.8
Assume the tension is described by a function $T(x)$ and the height is described by a function $y(x)$. Consider a small piece of the chain. Balancing forces in the horizontal direction gives

$$T(x)_x = T(x+dx)_x$$

Thus the horinzontal component of the tension is constant. Balancing forces in the vertical direction gives

$$T(x)_y = T(x+dx)_y + \lambda g\sqrt{1+y'(x)^2}dx$$
$$T'(x)_y = \lambda g\sqrt{1+y'(x)^2}$$

But because the tension must always point along the chain, we have

$$\frac{T(x)_y}{T(x)_x} = y'(x)$$

$$T(x)_y = y'(x)T(x)_x$$

Because $T(x)_x$ is constant

$$T'(x)_y = y''(x)T(x)_x$$

Setting $C = \frac{\lambda g}{T(x)_x}$ we have

$$y''(x) = C\sqrt{1+y'(x)^2}$$
$$y''(x)^2 = C^2 + C^2y'(x)^2$$

Letting $z(x) = y'(x)$

$$z'(x)^2 = C^2 + C^2z(x)^2$$

With a change of notation

$$\frac{dz}{dx} = \sqrt{C^2+C^2z^2}$$
$$\frac{dz}{\sqrt{C^2+C^2z^2}} = dx$$
$$\int \frac{dz}{\sqrt{1+z^2}} = \int C\,dx$$

With n as the constant of integration

$$\sinh^{-1}(z) = xC + n$$

$$z = \sinh(xC+n)$$

Substituting back

$$dy = \sinh(xC+n)dx$$

With h as another constant of integration

$$y = \frac{\cosh(xC+n)}{C} + hdx$$

However, n is always zero if we define x=0 as the lowest point of the chain, because

$$y'(0) = 0$$

$$\sinh(0+n) = 0$$

$$n = 0$$

So we have

$$y = \frac{1}{C}\cosh(xC) + h$$

where $C = \frac{\lambda g}{T(x)_x}$.

b) We have from the given quantities

$$x_f - x_i = d$$

$$y(x_f) - y(x_i) = \lambda$$

and

$$\int_{x_i}^{x_f} \sqrt{1 + \sinh(xC)^2}dx = \ell$$

Using the hyperbolic trig identity

$$\cosh(x)^2 - \sinh(x)^2 = 1$$

this is equivalent to

$$\int_{x_i}^{x_f} \sqrt{\cosh(xC)^2}dx = \ell$$

$$\int_{x_i}^{x_f} \cosh(xC)dx = \ell$$

$$\frac{1}{C}(\sinh(x_f C) - \sinh(x_i C)) = \ell$$

and substituting the other equation is equivalent to

$$\frac{1}{C}(\cosh(x_f C) - \cosh(x_i C)) = \lambda$$

Squaring both we get

$$\frac{1}{C^2}(\sinh(x_f C)^2 - 2\sinh(x_i C)\sinh(x_f C) + \sinh(x_i C)^2) = \ell^2$$

$$\frac{1}{C^2}(\cosh(x_f C)^2 - 2\cosh(x_i C)\cosh(x_f C) + \cosh(x_i C)^2) = \lambda^2$$

Subtracting

$$\frac{1}{C^2}(-1 + 2(\cosh(x_i C)\cosh(x_f C) - \sinh(x_i C)\sinh(x_f C)) - 1) = \ell^2 - \lambda^2$$

By the identity $\cosh(x)\cosh(y) - \sinh(x)\sinh(y) = \cosh(x-y)$, this is

$$(2\cosh(Cd) - 2) = C^2(\ell^2 - \lambda^2)$$

which is the required equation.

2.9

The support force F is given by the equation

$$2F\sin(\arctan(\sinh(Cd))) = \lambda g \ell$$

$$2F\tanh(Cd) = \lambda g$$

From 2.8 we have

$$\frac{1}{C}(\sinh(dC) - \sinh(-dC)) = \ell$$

Thus

$$F = \frac{\lambda g \sinh(dC)}{C\tanh(dC)}$$

$$F = \frac{\lambda g \cosh(dC)}{C}$$

This is minimized when $\frac{dF}{dC} = 0$

$$\frac{\lambda g(Cd\sinh(Cd) - \cosh(Cd))}{C^2} = 0$$

$$Cd\sinh(Cd) - \cosh(Cd) = 0$$

$$Cd\sinh(Cd) = \cosh(Cd)$$

$$\tanh(Cd) = \frac{1}{Cd}$$

Numerically solving this we get

$$Cd \approx 1.19968...$$

Denoting that number with α we have

$$C = \frac{\alpha}{d}$$

$$\frac{2d\sinh(\alpha)}{\alpha} = \ell$$

2.10

Honestly I have still not managed to solve this problem :/

2.11

Consider a number n. We know for every natural number m such that $m < n$ that a force F applied at a distance ℓ is equivalent to a force mF applied at a distance ℓ/m. Now consider a stick with forces F applied upwards at both ends 0 and ℓ and a force $2F/(n-1)$ applied at every point

$$\frac{\ell}{n}, \frac{2\ell}{n}, \frac{3\ell}{n} ... \frac{(n-1)\ell}{n}$$

where distances are measured from the left end. The stick does not translate because the net force is zero and it does not rotate by symmetry. Now consider the stick to have a pivot at the left end. Because we know for every natural number m such that $m < n$ that a force F applied at a distance ℓ is equivalent to a force mF applied at a distance ℓ/m, we can simplify the forces acting downward to be equivalent to be a single force acting downward at a distance $\frac{\ell}{n}$ from the left end with a magnitude

$$\sum_{k=1}^{n-1} \frac{2Fk}{n-1}$$

$$\frac{2F(n-1)n}{2(n-1)}$$

$$Fn$$

Thus, we see that a force F at a distance ℓ is equivalent to a force Fn at a distance $\frac{\ell}{n}$. By induction, this is true for all natural numbers.

2.12

If the tension does not point directly along the rope, there is a nonzero unbalanced torque, which the string cannot support because it is completely flexible.

2.13

This system is unsolvable. There are three unknowns and two equations.

2.14

Let us define L as the distance between the bottom ends of the two sticks and λ as their mass density. The left stick has a length of $L\sin(\theta)$. The torque exerted by gravity on the left stick is

$$\int_0^{L\sin(\theta)} \lambda g \sin(\theta) x \, dx$$

$$\frac{L^2 \sin(\theta)^3 \lambda g}{2}$$

Thus the normal force between the sticks must satisfy

$$\frac{L^2 \sin(\theta)^3 \lambda g}{2} = L\sin(\theta) F_{norm}$$

$$F_{norm} = \frac{L\sin(\theta)^2 \lambda g}{2}$$

Similarly, the torque due to gravity on the right stick is

$$\int_0^{L\cos(\theta)} \lambda g \cos(\theta) x \, dx$$

$$\frac{L^2 \cos(\theta)^3 \lambda g}{2}$$

Thus the friction force must be

$$F_{fric} = \frac{L\cos(\theta)^2 \lambda g}{2}$$

Thus to not fall we must have

$$F_{fric} \leq \mu F_{norm}$$

$$\frac{L\cos(\theta)^2 \lambda g}{2} \leq fracL\mu \sin(\theta)^2 \lambda g2$$

$$\cos(\theta)^2 \leq \mu \sin(\theta)^2$$

$$\frac{1}{\mu} \leq \tan(\theta)^2$$

$$\theta \geq \arctan\sqrt{\frac{1}{\mu}}$$

2.15

The torque on the ladder is

$$\int_0^{L} \frac{Mg\cos(\theta)x}{L} \, dx$$

$$\frac{Mg\cos(\theta)L}{2}$$

Thus the normal force must satisfy

$$F_{norm}\frac{\ell}{\tan(\theta)} = \frac{Mg\cos(\theta)L}{2}$$

$$F_{norm} = \frac{Mg\sin(\theta)L}{2\ell}$$

2.16

Let the stick have a mass density $\lambda(x)$. Let the stick be cut off at the point x_0. Thus the torques must balance so we have

$$\int_{x_0}^{x_0+\ell}(x_0+\ell-x)\lambda(x)g\,dx = \int_{x_0+\ell}^{\infty}(x-x_0-\ell)\lambda(x)g\,dx$$

$$\int_{x_0}^{x_0+\ell}(x_0+\ell)\lambda(x)-x\lambda(x)\,dx = \int_{x_0+\ell}^{\infty}x\lambda(x)-(x_0+\ell)\lambda(x)\,dx$$

$$\int_{x_0}^{\infty}(x_0+\ell)\lambda(x)-x\lambda(x)\,dx = 0$$

$$\int_{x_0}^{\infty}(x_0+\ell)\lambda(x)g\,dx = \int_{x_0}^{\infty}x\lambda(x)g\,dx$$

Unfinished because i am dumb

2.17

a) Balancing forces in the horizontal direction gives

$$T\cos(\theta) = F_{frict}$$

and balancing torques gives

$$rT = RF_{frict}$$

Thus

$$\frac{rT}{R} = T\cos(\theta)$$

$$\theta = \arccos(\frac{r}{R})$$

b)

$$T \leq \frac{R\mu F_{norm}}{r}$$

Thus at the maximum T we have

$$T = \frac{R\mu(mg - T\sin(\theta)}{r}$$

$$Tr + TR\mu\sin(\theta) = Rmg\mu$$

$$T = \frac{Rmg\mu}{r + R\mu\sin(\theta)}$$

Using $r = R\cos(\theta)$ this becomes

$$T = \frac{Rmg\mu}{R\cos(\theta) + R\mu\sin(\theta)}$$

$$T = \frac{mg\mu}{\cos(\theta) + \mu\sin(\theta)}$$

c)

$$\cos(\theta) = \frac{r}{R}$$

$$\sin^2(\theta) + \frac{r^2}{R^2} = 1$$

$$\sin(\theta) = \sqrt{1 - \frac{r^2}{R^2}}$$

$$T = \frac{mg\mu}{\frac{r}{R} + \mu\sqrt{1 - \frac{r^2}{R^2}}}$$

Simplifying

$$T = \frac{Rmg\mu}{r + \mu\sqrt{R^2 - r^2}}$$

This is minimized when the derivative with respect to r is zero so

$$\frac{\delta}{\delta r}(\frac{Rmg\mu}{r + \mu\sqrt{R^2 - r^2}}) = 0$$

$$-\frac{gmR\mu(1 - \frac{r\mu}{\sqrt{R^2 - r^2}})}{(\mu\sqrt{R^2 - r^2} + r)^2} = 0$$

$$gmR\mu(\frac{r\mu}{\sqrt{R^2 - r^2}} - 1) = 0$$

$$\sqrt{R^2 - r^2} = r\mu$$

$$R^2 - r^2 = r^2\mu^2$$

$$R^2 = r^2(1 + \mu^2)$$

$$r = \sqrt{\frac{R^2}{1 + \mu^2}}$$

$$r = \frac{R}{\sqrt{1 + \mu^2}}$$

2.18
The torque on the stick from gravity is

$$\frac{L^2\cos(\theta)\rho g}{2}$$

By similar triangles the height of the point of contact off the ground is

$$R(1 + \cos(\theta))$$

Thus on the other triangle

$$\sin(\theta) = \frac{R(1+\cos(\theta))}{L}$$

so L is

$$\frac{R(1+\cos(\theta))}{\sin(\theta)}$$

Thus the torque becomes

$$\frac{LR(1+\cos(\theta))\cos(\theta)\rho g}{2\sin(\theta)}$$

so the normal force must be

$$\frac{R(1+\cos(\theta))\cos(\theta)\rho g}{2\sin(\theta)}$$

The horizontal component of the normal force is

$$\frac{R(1+\cos(\theta))\cos(\theta)\sin(\theta)\rho g}{2\sin(\theta)}$$

$$\frac{R(1+\cos(\theta))\cos(\theta)\rho g}{2}$$

But the torques must also balance, so the frictional force between the stick and the circle must be the same as between the circle and the ground. Thus we have

$$\frac{R(1+\cos(\theta))\cos(\theta)\rho g}{2} = F_{frict}(1+\cos(\theta))$$

So the frictional force is

$$\frac{R\cos(\theta)\rho g}{2}$$

2.19

Denote the torque due to gravity on any one stick τ, the length of any one stick L, and the distance between the bottom of any stick and its point of contact with the previous circle ℓ. The normal force between the first stick and circle is

$$\frac{\tau}{L}$$

, which means this is also the normal force between the first circle and the second stick. Balancing the torques on the second stick gives

$$\frac{\ell\tau}{L} + \tau = F_{norm}(2)L$$

$$F_{norm}(2) = \frac{\tau}{L}(1+\frac{\ell}{L})$$

In general

$$\ell F_{norm}(n) + \tau = F_{norm}(n+1)L$$

$$F_{norm}(n+1) = \frac{\ell F_{norm}(n) + \tau}{L}$$

In the limit assuming this goes to a constant value we have

$$F_{norm}(\infty) = \frac{\ell F_{norm}(\infty) + \tau}{L}$$

$$F_{norm}(\infty)L = \ell F_{norm}(\infty) + \tau$$
$$F_{norm}(\infty) = \frac{\tau}{L-\ell}$$

By the reasoning of problem 2.18,

$$\tau = \frac{L^2\cos(\theta)\rho g}{2}$$
$$L = \frac{R(1+\cos(\theta))}{\sin(\theta)}$$

By *mysterious geometry*

$$\ell = R\tan(\frac{\theta}{2})$$
$$F_{norm}(\infty) = \frac{\frac{L^2\cos(\theta)\rho g}{2}}{\frac{R(1+\cos(\theta))}{\sin(\theta)} - R\tan(\frac{\theta}{2})}$$
$$F_{norm}(\infty) = \frac{(\frac{R(1+\cos(\theta))}{\sin(\theta)})^2\frac{\cos(\theta)\rho g}{2}}{\frac{R(1+\cos(\theta))}{\sin(\theta)} - \frac{R(1-\cos(\theta))}{\sin(\theta)}}$$
$$F_{norm}(\infty) = \frac{\frac{\cos(\theta)\rho g R^2(1+\cos(\theta))^2}{2\sin^2(\theta)}}{\frac{2R\cos(\theta)}{\sin(\theta)}}$$
$$F_{norm}(\infty) = \frac{\rho g R(1+\cos(\theta))^2}{4\sin(\theta)}$$

Note: this is not the answer the answer key gives, but somehow they're exactly the same by *mysterious trig identities.*

3.1

In all honesty i have this equation memorized

$$a_1 = a_2 = g\frac{m_2-m_1}{m_1+m_2}$$
$$T = g\frac{2m_1m_2}{m_1+m_2}$$

:)

3.2

Equations:

$$T_1 - m_1g = m_1a_1$$
$$T_2 - m_2g = m_2a_2$$
$$T_2 - m_3g = m_3a_3$$
$$T_1 = 2T_2$$
$$2a_1 + a_2 = -a_3$$

Substitute 4 and 5 into 1, 2 and 3:

$$2T_2 - m_1a_1 = m_1g$$
$$T_2 - m_2a_2 = m_2g$$

$$T_2 + (2a_1 + a_2)(m_3) = m_3 g$$

Subtract 7 from 8:

$$(2a_1 + a_2)(m_3) + m_2 a_2 = (m_3 - m_2)g$$

$$a_1 = \frac{(m_3 - m_2)g - (m_3 + m_2)a_2}{2m_3}$$

Substitute 10 into 6:

$$2T_2 - m_1 \frac{(m_3 - m_2)g - (m_3 + m_2)a_2}{2m_3} = m_1 g$$

Simplify:

$$4T_2 m_3 - m_1(m_3 - m_2)g - m_1(m_3 + m_2)a_2 = 2m_3 m_1 g$$

$$4T_2 m_3 = m_1(m_3 - m_2)g + m_1(m_3 + m_2)a_2 + 2m_3 m_1 g$$

$$4T_2 m_3 = m_1(3m_3 - m_2)g + m_1(m_3 + m_2)a_2$$

$$T_2 = \frac{m_1(3m_3 - m_2)g + m_1(m_3 + m_2)a_2}{4m_3}$$

Substitute into 7:

$$\frac{m_1(3m_3 - m_2)g + m_1(m_3 + m_2)a_2}{4m_3} - m_2 a_2 = m_2 g$$

$$m_1(3m_3 - m_2)g + m_1(m_3 + m_2)a_2 - 4m_3 m_2 a_2 = 4m_3 m_2 g$$

$$a_2 = \frac{4m_3 m_2 g - m_1(3m_3 - m_2)g}{m_1(m_3 + m_2) - 4m_3 m_2}$$

3.3

uh wtf

3.4

For the two on the ends,

$$T - mg = ma_1$$

For the ones in the middle,

$$2T - mg = ma_2$$

And conservation of string gives

$$-Na_2 = a_1$$

Solving

$$2T - 2mg = 2ma_1$$

$$mg = m(a_2 - 2a_1)$$

$$g = a_2 + 2Na_2$$

$$a_2 = \frac{g}{1 + 2N}$$

$$a_1 = \frac{-Ng}{1 + 2N}$$

3.5

$$2T - m_i g = m_i a_i$$

$$\sum_{i=1}^{N} a_i = 0$$

$$\sum_{i=1}^{N} \frac{2T - m_i g}{m_i} = 0$$

$$\sum_{i=1}^{N} \frac{2T}{m_i} = Ng$$

$$\sum_{i=1}^{N} \frac{1}{m_i} = \frac{Ng}{2T}$$

Define the reduced mass

$$M := \frac{1}{\sum_{i=1}^{N} \frac{1}{m_i}}$$

Then

$$T = \frac{NMg}{2}$$

$$NMg - m_i g = m_i a_i$$

$$a_i = \frac{NMg - m_i g}{m_i}$$

3.6

a)

$$\Delta x = \frac{at^2}{2}$$

$$mg\sin(\theta) = ma$$

$$\Delta x = \frac{g\sin(\theta)t^2}{2}$$

$$\Delta x_h = \frac{g\sin(\theta)\cos(\theta)t^2}{2}$$

$$\frac{d}{d\theta}(x_h) = \frac{gt^2}{2}(\cos^2(\theta) - \sin^2(\theta)) = 0$$

$$\cos^2(\theta) = \sin^2(\theta)$$

$$\cos(\theta) = \sin(\theta)$$

$$\theta = \frac{\pi}{2}$$

b)

$$\Delta x = \frac{at^2}{2}$$

$$mg\sin(\theta) - \mu mg\cos(\theta) = ma$$

$$\Delta x = \frac{g(\sin(\theta) - \mu\cos(\theta))t^2}{2}$$

$$\Delta x_h = \frac{g(\sin(\theta) - \mu\cos(\theta))\cos(\theta)t^2}{2}$$

$$\frac{d}{d\theta}(x_h) = \frac{gt^2}{2}((\cos(\theta) + \mu\sin(\theta))\cos(\theta) - (\sin(\theta) - \mu\cos(\theta))\sin(\theta)) = 0$$

$$\cos^2(\theta) + 2\mu\cos(\theta)\sin(\theta) - \sin^2(\theta) = 0$$

$$\cos(2\theta) + \mu\sin(2\theta) = 0$$

$$\tan(2\theta) = -\frac{1}{\mu}$$

$$2\theta = \arctan(-\frac{1}{\mu})$$

$$\theta = \frac{\arctan(-\frac{1}{\mu})}{2}$$

3.7
At the beginning, $v_\perp$ is V and $v_\parallel$ is 0. The total force of friction is

$$\mu mg\cos(\theta)$$

$$mg\sin(\theta)$$

$$F_{frict\perp} = mg\sin(\theta)\sin(\arctan(\frac{v_\perp}{v_\parallel}))$$

$$F_{frict\perp} = mg\sin(\theta)\frac{\frac{v_\perp}{v_\parallel}}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}}$$

$$F_{frict\parallel} = mg\sin(\theta)\cos(\arctan(\frac{v_\perp}{v_\parallel}))$$

$$F_{frict\parallel} = mg\sin(\theta)\frac{1}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}}$$

$$\frac{d}{dt}(v_\perp) = -g\sin(\theta)\frac{\frac{v_\perp}{v_\parallel}}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}}$$

$$\frac{d}{dt}(v_\parallel) = g\sin(\theta)(1 - \frac{1}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}}) = g\sin(\theta)\frac{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1} - 1}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}}$$

$$\frac{d}{dt}(v) = \frac{d}{dt}(\sqrt{v_\perp^2 + v_\parallel^2}) = \frac{v_\perp\frac{d}{dt}(v_\perp) + v_\parallel\frac{d}{dt}(v_\parallel)}{\sqrt{v_\perp^2 + v_\parallel^2}}$$

$$\frac{d}{dt}(v) = \frac{g\sin(\theta)}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}\sqrt{v_\perp^2 + v_\parallel^2}}(v_\parallel(\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1} - 1) - v_\perp\frac{v_\perp}{v_\parallel})$$

$$\frac{d}{dt}(v) = \frac{g\sin(\theta)v_\parallel}{\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1}\sqrt{v_\perp^2 + v_\parallel^2}}(\sqrt{\frac{v_\perp}{v_\parallel}^2 + 1} - (1 + \frac{v_\perp}{v_\parallel}^2))$$

$$\frac{d}{dt}(v) = \frac{g\sin(\theta)v_{\parallel}(1 - \sqrt{\frac{v_{\perp}}{v_{\parallel}}^2 + 1})}{\sqrt{v_{\perp}^2 + v_{\parallel}^2}}$$

$$\frac{d}{dt}(v) = \frac{g\sin(\theta)(1 - \sqrt{\frac{v_{\perp}}{v_{\parallel}}^2 + 1})}{\sqrt{\frac{v_{\perp}}{v_{\parallel}}^2 + 1}} = -\frac{d}{dt}(v_{\parallel})$$

$$v + v_{\parallel} = c$$

$$V + 0 = c$$

$$c = V$$

$$v + v_{\parallel} = V$$

After a very long time all of the block's velocity is in the direction parallel to the plane so

$$2v_{\parallel} = V$$

$$v_{\parallel} = \frac{V}{2}$$

3.8

Define F_{norm} as the force between the block and the plane. Equations of motion:

$$F_{norm}\sin(\theta) = -Ma_{px}$$

$$F_{norm}\sin(\theta) = ma_{bx}$$

$$F_{norm}\cos(\theta) - mg = ma_{by}$$

$$\frac{-a_{by}}{a_{bx} - a_{px}} = \tan(\theta)$$

$$\frac{\frac{F_{norm}\cos(\theta) - mg}{m}}{\frac{F_{norm}\sin(\theta)}{m} + \frac{F_{norm}\sin(\theta)}{M}} = \tan(\theta)$$

$$\frac{\frac{F_{norm}\cos(\theta) - mg}{m}}{\frac{F_{norm}\sin(\theta)(M+m)}{mM}} = \tan(\theta)$$

$$\frac{F_{norm}\cos(\theta)M - mMg}{F_{norm}\sin(\theta)(M+m)} = \tan(\theta)$$

$$F_{norm}(\cos(\theta)M - \sin(\theta)(M+m)\tan(\theta)) = mMg$$

$$F_{norm} = \frac{mMg}{(\cos(\theta)M - \sin(\theta)(M+m)\tan(\theta))}$$

$$a_{px} = -\frac{mg\sin(\theta)}{(\cos(\theta)M - \sin(\theta)(M+m)\tan(\theta))}$$

3.9

$$F(t) = ma_0e^{-bt}$$

$$x''(t) = a_0e^{-bt}$$

$$x'(t) = \frac{-a_0 e^{-bt}}{b} + \alpha$$
$$0 = \frac{-a_0}{b} + \alpha$$
$$x'(t) = \frac{-a_0 e^{-bt}}{b} + \frac{a_0}{b}$$
$$x(t) = \frac{a_0 e^{-bt}}{b^2} + \frac{a_0 t}{b} + \beta$$
$$0 = \frac{a_0}{b^2} + \beta$$
$$x(t) = \frac{a_0 e^{-bt}}{b^2} + \frac{a_0 t}{b} - \frac{a_0}{b^2}$$

3.10

$$F(x) = -kx$$
$$mv\frac{dv}{dx} = -kx$$
$$mv\,dv = -kx\,dx$$
$$\frac{mv^2}{2} = \frac{-kx^2}{2}$$
$$v = \sqrt{\frac{-kx^2}{m}}$$
$$\sqrt{\frac{-m}{kx^2}}\,dx = dt$$
$$(\sqrt{\frac{-m}{k}})\frac{1}{x}\,dx = dt$$
$$(\sqrt{\frac{-m}{k}})\ln(x) = t$$
$$\ln(x) = \sqrt{\frac{-t^2k}{m}}$$
$$x = x_0 e^{t\sqrt{\frac{-k}{m}}}$$
$$x = x_0\cos(t\sqrt{\frac{k}{m}})$$

3.11

$$F = y(t)\lambda g$$
$$a = \frac{y(t)\lambda g}{\ell\lambda} = \frac{y(t)g}{\ell}$$
$$y''(t) = \frac{g}{\ell}y(t)$$

$$y(t) = \alpha e^{t\sqrt{\frac{g}{\ell}}} + \alpha e^{-t\sqrt{\frac{g}{\ell}}}$$

$$y'(t) = \sqrt{\frac{g}{\ell}}\alpha e^{t\sqrt{\frac{g}{\ell}}} + \sqrt{\frac{g}{\ell}}\beta e^{-t\sqrt{\frac{g}{\ell}}}$$

$$\alpha + \beta = y_0$$

$$\sqrt{\frac{g}{\ell}}\alpha - \sqrt{\frac{g}{\ell}}\beta = 0$$

$$\alpha = \beta = \frac{y+0}{2}$$

$$y(t) = y_0 \frac{e^{\sqrt{\frac{g}{\ell}}t} + e^{-\sqrt{\frac{g}{\ell}}t}}{2} = y_0 \cosh(\sqrt{\frac{g}{\ell}}t)$$

3.12

Note to self: finish problem

$$F(v) = mg - m\alpha v$$

$$g - \alpha v = \frac{dv}{dt}$$

$$dt = \frac{dv}{g - \alpha v}$$

$$t = \frac{-\ln(g - \alpha v)}{\alpha} + c$$

$$0 = \frac{-\ln(g - \alpha v_0)}{\alpha} + c$$

$$c\alpha = \ln(g - \alpha v_0)$$

$$c = \frac{\ln(g - \alpha v_0)}{\alpha}$$

$$-t\alpha = \ln(g - \alpha v) - \ln(g - \alpha v_0)$$

$$e^{-t\alpha} = \frac{g - \alpha v}{g - \alpha v_0}$$

$$e^{-t\alpha}(g - \alpha v_0) = g - \alpha v$$

$$\frac{g - e^{-t\alpha}(g - \alpha v_0)}{\alpha} = v$$

$$\frac{g - e^{-t\alpha}g + e^{-t\alpha}\alpha v_0}{\alpha} = v$$

$$x = \frac{gt + \frac{e^{-t\alpha}g - e^{-t\alpha}\alpha v_0}{\alpha}}{\alpha} + c$$

$$x = \frac{\alpha gt + e^{-t\alpha}g - e^{-t\alpha}\alpha v_0}{\alpha^2} + c$$

$$0 = \frac{g - \alpha v_0}{\alpha^2} + c$$

$$x = \frac{\alpha gt + e^{-t\alpha}g - e^{-t\alpha}\alpha v_0 - g + \alpha v_0}{\alpha^2}$$

$$x = \frac{\alpha g t + e^{-t\alpha}(g - \alpha v_0) - (g - \alpha v_0)}{\alpha^2}$$
$$x = \frac{\alpha g t + (e^{-t\alpha} - 1)(g - \alpha v_0)}{\alpha^2}$$

3.13
a)

$$\tau = mg\ell \sin(\theta) \approx mg\ell\theta$$
$$m\ell^2\theta''(t) = mg\ell\theta(t)$$
$$\theta''(t) = \frac{g}{\ell}\theta(t)$$

The general solution is

$$\theta(t) = \alpha e^{t\sqrt{\frac{g}{\ell}}} + \beta e^{-t\sqrt{\frac{g}{\ell}}}$$

Using the initial conditions

$$\theta_0 = \alpha + \beta$$
$$\theta'(t) = \sqrt{\frac{g}{\ell}}(\alpha e^{t\sqrt{\frac{g}{\ell}}} - \beta e^{-t\sqrt{\frac{g}{\ell}}})$$
$$\omega_0 = \sqrt{\frac{g}{\ell}}(\alpha - \beta)$$
$$\alpha = \theta_0 - \beta$$
$$\omega_0 = \sqrt{\frac{g}{\ell}}(\theta_0 - 2\beta)$$
$$\beta = \frac{1}{2}(\theta_0 - \omega_0\sqrt{\frac{\ell}{g}})$$
$$\alpha = \frac{1}{2}(\theta_0 + \omega_0\sqrt{\frac{\ell}{g}})$$
$$\theta(t) = \frac{1}{2}(\theta_0 + \omega_0\sqrt{\frac{\ell}{g}})e^{t\sqrt{\frac{g}{\ell}}} + \frac{1}{2}(\theta_0 - \omega_0\sqrt{\frac{\ell}{g}})e^{-t\sqrt{\frac{g}{\ell}}}$$

b)

$$(\ell\theta_0)(m\ell\omega_0) \geq \hbar$$
$$\omega_0 \geq \frac{\hbar}{m\ell^2\theta_0}$$

Because α and β are both so small, the negative exponential term is neglibigle, so we have

$$\theta(t) = \frac{1}{2}(\theta_0 + \frac{\hbar}{m\ell^2\theta_0}\sqrt{\frac{\ell}{g}})e^{t\sqrt{\frac{g}{\ell}}}$$

So we must maximize the coefficient

$$\theta_0 + \frac{\hbar}{m\ell^2\theta_0}\sqrt{\frac{\ell}{g}}$$

Thus

$$1 - \frac{\hbar}{m\ell^2\theta_0^2}\sqrt{\frac{\ell}{g}} = 0$$

$$\frac{\hbar}{m\ell^2}\sqrt{\frac{\ell}{g}} = \theta_0^2$$

$$\sqrt{\frac{\hbar}{m\ell^2}\sqrt{\frac{\ell}{g}}} = \theta_0$$

$$1 = \sqrt{\frac{\hbar}{m\ell^2}\sqrt{\frac{\ell}{g}}}e^{t\sqrt{\frac{g}{\ell}}}$$

$$\sqrt{\frac{m\ell^2}{\hbar}\sqrt{\frac{g}{\ell}}} = e^{t\sqrt{\frac{g}{\ell}}}$$

$$\ln(\sqrt{\frac{m\ell^2}{\hbar}\sqrt{\frac{g}{\ell}}}) = t\sqrt{\frac{g}{\ell}}$$

$$t = \frac{\ln(\frac{m^2\ell^3 g}{\hbar^2})}{4\sqrt{\frac{g}{\ell}}}$$

$$t \approx 3.5\, sec$$

3.14

$$x = v\cos(\theta)t$$

$$y = v\sin(\theta)t - \frac{1}{2}gt^2$$

$$t = \frac{x}{v\cos(\theta)}$$

$$y = v\sin(\theta)\frac{x}{v\cos(\theta)} - \frac{1}{2}g\frac{x^2}{v^2\cos(\theta)^2}$$

$$y = x\tan(\theta) - \frac{gx^2}{2v^2\cos(\theta)^2}$$

$$0 = x\tan(\theta) - \frac{gx^2}{2v^2\cos(\theta)^2}$$

$$\tan(\theta) = \frac{gx}{2v^2\cos(\theta)^2}$$

$$\frac{2v^2\sin(\theta)\cos(\theta)}{g} = x_f$$

$$\int_0^{x_f} x\tan(\theta) - \frac{gx^2}{2v^2\cos(\theta)^2}\, dx$$

$$|\frac{x^2\tan(\theta)}{2} - \frac{gx^3}{6v^2\cos(\theta)^2}|_0^{x_f}$$

$$\frac{\tan(\theta)}{2}(\frac{2v^2\sin(\theta)\cos(\theta)}{g})^2 - \frac{g}{6v^2\cos(\theta)^2}(\frac{2v^2\sin(\theta)\cos(\theta)}{g})^3$$

$$\frac{2\sin^3(\theta)\cos(\theta)v^4}{g^2} - \frac{4\sin^3(\theta)\cos(\theta)}{3}$$

$$\frac{2\sin^3(\theta)\cos(\theta)v^4}{3g^2}$$

$$\frac{d}{dx}(\frac{2\sin^3(\theta)\cos(\theta)v^4}{3g^2}) = 0$$

$$(3\sin^2(\theta)\cos^2(\theta) - \sin^4(\theta))\frac{2v^4}{3g^2} = 0$$

$$3\sin^2(\theta)\cos^2(\theta) = \sin^4(\theta)$$

$$3\cos^2(\theta) = \sin^2(\theta)$$

$$\tan(\theta) = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

$$A_{max} = \frac{2\sin^3(\frac{\pi}{3})\cos(\frac{\pi}{3})v^4}{3g^2}$$

$$A_{max} = \frac{\sqrt{3}v^4}{8g^2}$$